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OPTIMUM PLASTIC DESIGN OF STRUCTURES UNDER COMBINED STRESSES†

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Abstract—In the framework of linear plastic design of a rather wide class of discrete structures, optimality conditions for optimum design are considered and cast in a form which gives rise to a significant new geometrical description. In this description the designing is viewed as a growing process of the structure, which is governed by a set of relations similar to those governing the plastic flowing process. A finite element-linear programming method is used, and allowance is made for design dependent mass forces and for some technological constraints.

1. INTRODUCTION

After the first major contribution to the optimization of plastic structures, due to Michell[1], many authors have dealt with this topic. An extensive bibliography is found in Massonet and Save[2], as well as in the critical survey of Sheu and Prager[3], where the right emphasis is put upon the connection between plastic limit design and linear programming. This connection has been stressed more and more in recent years, and generalizations have been conceived to incorporate instability[4] and serviceability[5, 6] aspects in the framework of the theory. A systematic discussion of plastic limit design of frames using linear programming has been given by Maier *et al.*[7], where four different mathematical formulations are set up and allowance is made for design dependent selfweight of structures.

Nonlinear plastic design has received some attention from various authors (see e.g. [8–10]). A class of nonlinear plastic design problems, with convex cost functions, were solved as statical problems in nonlinear elasticity with constitutive laws directly deduced from the cost function, viewed as the relevant complementary energy function[11]. This theory was extended by Prager and Shield[12] to arbitrary 1- or 2-dimensional structures. Later Chan[13] arrived at a mathematical programming formulation of the Marcal–Prager theory, also taking into account multiple sets of loads. Finally, Razvany and Adidam[14] obtained dual relations between the yield surface and the cost gradient surface (or between limit analysis and optimal limit design). Nonlinear plastic design has been also approached via dynamic programming (see e.g. [15–17]).

The finite element method has been intensively used in the optimum plastic design of structures, in connection with linear programming (see e.g. [18, 19]). In [18] the optimum design of plastic discs was treated, while in [19] the design in plane stresses was considered. The work quoted above in [7] assumes a discrete model with lumped deformability.

This paper is intended to treat, via finite element method–linear programming, the optimum limit design of a wide class of plastic structures, at least comprehensive of those treated in [7],

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[18] and [19], and to give optimality conditions in an apparently new geometrical form. A correspondence is established between the generalized stress space, where *conformity* conditions are to be satisfied, and the plastic deformation intensity space, where *uniformity* conditions are to be satisfied in turn. The optimality conditions appear to be a generalization of the *uniform energy dissipation principle* of Drucker and Shield [20], as well as the *cost gradient principle* of Marcal *et al.* [11, 12].

NOTATION

Capital bold-face letters are matrices, small bold face letters are vectors. **0** is a vector having only zero entries. The tilde posed upon a vector or a matrix signifies "transpose of". The meaning of the symbols used later will be given where they first appear.

2. HYPOTHESES

(a) The structure to be optimized is conceived as a discrete model, made up of n *finite elements.* The ith finite element has a dimension of measure l_i ($i = 1,2,\ldots,n$) (length for 1-, area for 2- and volume for 3-dimensional elements).

(b) An elemental rigid–plastic behaviour is assumed for every finite element. In the generalized stress space the yielding surface of the element is a polyhedron, which is possibly obtained by linearization of the actual smooth surface. A plastic resistance k_j and a yielding function φ_j are associated with the jth ($j = 1,2,\ldots,s$) face of the polyhedron. The s yielding functions $\varphi_j (j = 1,2,\ldots,s)$ of the ith element can be expressed in vectorial form [21, 22]:

$$\boldsymbol{\varphi}_i = \tilde{\mathbf{N}}_i \boldsymbol{\sigma}_i - \mathbf{k}_i, \quad (i = 1,2,\ldots,n)$$

where $\tilde{\boldsymbol{\varphi}}_i = [\varphi_{i,1}, \varphi_{i,2}, \ldots, \varphi_{i,s}]$, $\tilde{\mathbf{k}}_i = [k_{i,1}, k_{i,2}, \ldots, k_{i,s}]$, $\boldsymbol{\sigma}_i$ is the generalized stress vector, and $\mathbf{N}_i = [\mathbf{N}_{i,1}, \mathbf{N}_{i,2} \ldots, \mathbf{N}_{i,s}]$ is the matrix of the unit normals to the relevant faces of the polyhedron (see Fig. 2a). The n vectors $\boldsymbol{\varphi}_i$ ($i = 1,2,\ldots,n$) can be described by a single compact matrix expression:

$$\boldsymbol{\varphi} = \tilde{\mathbf{N}}\boldsymbol{\sigma} - \mathbf{k}, \tag{2.1}$$

where $\tilde{\boldsymbol{\varphi}} = [\tilde{\boldsymbol{\varphi}}_1, \tilde{\boldsymbol{\varphi}}_2, \ldots, \tilde{\boldsymbol{\varphi}}_n]$, $\tilde{\boldsymbol{\sigma}} = [\tilde{\boldsymbol{\sigma}}_1, \tilde{\boldsymbol{\sigma}}_2, \ldots, \tilde{\boldsymbol{\sigma}}_n]$, $\tilde{\mathbf{k}} = [\tilde{\mathbf{k}}_1, \tilde{\mathbf{k}}_2, \ldots, \tilde{\mathbf{k}}_n]$ and $\mathbf{N} = \mathrm{diag}\,[\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n]$.

(c) The unit cost of the element is to be specified. If $g_i (i = 1,2,\ldots,n)$ is the unit cost of the ith element, the overall cost Φ of the structure results:

$$\Phi = \sum_{i=1}^{n} g_i l_i = \tilde{\mathbf{l}}\mathbf{g}, \tag{2.2}$$

where $\tilde{\mathbf{l}} = [l_1, l_2, \ldots, l_n]$, and $\tilde{\mathbf{g}} = [g_1, g_2, \ldots, g_n]$.

(d) The unit cost of the ith element is a linear function of some, say $t \leq s$, design variables $r_h (h = 1,2,\ldots,t)$, i.e.:

$$g_i = a_i + \sum_{h=1}^{t} b_{ih}\, r_{ih} \tag{2.3,a}$$

$$= a_i + \mathbf{b}_i \mathbf{r}_i, \quad (i = 1,2,\ldots,n) \tag{2.3,b}$$

where $\tilde{\mathbf{r}}_i = [r_{i,1}, r_{i,2}, \ldots r_{i,t}]$, $\tilde{\mathbf{b}}_i = [b_{i,1}, b_{i,2}, \ldots, b_{i,t}]$. Putting now $\tilde{\mathbf{a}} = [a_1, a_2, \ldots, a_n]$, $\tilde{\mathbf{r}} = [\tilde{\mathbf{r}}_1,$

$\tilde{\mathbf{r}}_2, \ldots, \tilde{\mathbf{r}}_n]$, and $\mathbf{B} = \text{diag}[\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$, we can collect all the unit costs in a single compact expression:

$$\mathbf{g} = \mathbf{a} + \tilde{\mathbf{B}}\mathbf{r}. \tag{2.4}$$

The matrix $\mathbf{B}$ is the *specific cost* matrix (i.e. the entry b_{ih} of it means the cost per unit dimension of the ith element and per unit value of the relevant design variable r_{ih}).

The true relation between the unit cost and the design variables is often nonlinear; nevertheless linearity is a common hypothesis of many researches on optimum limit design and it permits one, in the nonlinear cases, to obtain good information about the true optimum structure.

(e) Displacements are infinitesimal, and geometrical effects on equilibrium equations are disregarded.

(f) Selfweight of the structure is taken into account along with some constraints of technological character, as in [7, 18].

(g) A continuous set of elements, of different shape and size, is available for a given node layout.

3. FUNDAMENTAL RELATIONS

Let $\dot{\boldsymbol{\epsilon}}$ be the strain rate vector, whose entries are arranged so that the matrix product $\tilde{\boldsymbol{\sigma}}\,\dot{\boldsymbol{\epsilon}}$ is the proper expression of the work rate; moreover let $\dot{\mathbf{d}}$ be the velocity vector of the application points of the loads $\mathbf{f}$, and $\dot{\mathbf{u}}$ the lagrangian velocity vector. For the *compatibility* the following equations are to be satisfied:

$$\dot{\boldsymbol{\epsilon}} = \mathbf{C}\dot{\mathbf{u}};\ \dot{\mathbf{d}} = \mathbf{C}^*\dot{\mathbf{u}}; \tag{3.1a,b}$$

where $\mathbf{C}$ and $\mathbf{C}^*$ are the compatibility matrices, which are dependent only on the node layout and on the load distribution.

The *equilibrium* equations are written:

$$\tilde{\mathbf{C}}\boldsymbol{\sigma} = \mathbf{p}, \text{ with } \mathbf{p} = \tilde{\mathbf{C}}^*\mathbf{f}. \tag{3.2a,b}$$

Conformity of stress and strain rates (i.e. the set of the consitutive relations for rigid–plastic elements) is described, following Maier[21, 22], as a *linear flow-law*:

$$\boldsymbol{\varphi} = \tilde{\mathbf{N}}\boldsymbol{\sigma} - \mathbf{k} \leq \mathbf{0}, \tag{3.3}$$

$$\dot{\boldsymbol{\epsilon}} = \mathbf{N}\dot{\boldsymbol{\lambda}};\ \dot{\boldsymbol{\lambda}} \geq \mathbf{0};\ \tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{\lambda}} = 0, \tag{3.4a–c}$$

where $\dot{\boldsymbol{\lambda}}$ is the ns-vector of the activation coefficients or deformation rate intensities. Since $\boldsymbol{\varphi}$ and $\dot{\boldsymbol{\lambda}}$ are sign constrained, the orthogonality condition (3.4,c) applies to each component $\varphi_j\dot{\lambda}_j$ ($j = 1,2,\ldots, ns$) and φ_j, $\dot{\lambda}_j$ cannot both be different from zero.

A design is completely described by the *design variable* vector $\mathbf{r}(\mathbf{r} \geq \mathbf{0})$. The plastic resistance vector $\mathbf{k}$ is defined by the matrix relation:

$$\mathbf{k} = \mathbf{V}\mathbf{r}, \tag{3.5}$$

where $\mathbf{V}$ is a $ns \times nt$-matrix with *nonnegative* entries, and it depends on the nature of the design variables. This matrix can be written $\mathbf{V} = \text{diag}[\mathbf{V}_1, \mathbf{V}_2, \ldots. \mathbf{V}_n]$, and the $s \times t$-matrices $\mathbf{V}_j$ enter into the following set of relations, which are equivalent to (3.5):

$$\mathbf{k}_i = \mathbf{V}_i \mathbf{r}_i, \quad (i = 1,2,\ldots,n). \tag{3.6}$$

When the vector **r** increases, there is a growth of all or some elements of the structure and an expansion of the relevant yielding polyhedra. A process like this will be called "design growing process", when it produces an optimum design. If $\mathbf{V} = \mathbf{I}$ (identity matrix), the plastic resistances **k** are design variables themselves, and in the design growing process the faces of the yielding polyhedra translate (without rotation) each other independently. If all the $\mathbf{V}_i$ are vectors (i.e. if $t = 1$), there is only one design variable for each finite element, and in the design growing process the yielding polyhedra expand in such a way that they remain similar to themselves. We have the intermediate situation when $t > 1$. A matrix $\mathbf{V} \neq \mathbf{I}$ implies some constraints on the free expansion of the yielding polyhedra (and then on the possible shapes of the elements).

Let us assume, for example, the polygonal line of Fig. 1 as a yield line relative to the rectangular cross section of a prismatic element, which is loaded by a bending moment M and a normal stress N. It is a linear approximation of the true parabolic yield line. If we take:

$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \end{bmatrix}; \quad \mathbf{V} = \begin{bmatrix} 1 & 0 \\ a & b \\ 0 & 1 \\ a & b \\ 1 & 0 \\ a & b \\ 0 & 1 \\ a & b \end{bmatrix}; \quad \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} M_L \\ N_L \end{bmatrix}$$

(with M_L, N_L simple yielding generalized stresses, and a, b, proper coefficients), the edges of the yield polygon expand remaining parallel to themselves during the design growing process. Whatever the optimum yield polygon may finally be, it is always possible, within some limits, to find a rectangular section with simple yield stresses which are equal to the optimum values of the design variables (and with a true yield line linearly approximated by the optimum yield polygon).

The *external forces* **f** are to be considered as the sum of the fixed service loads $\mathbf{f}_a$ and the mass forces (or self-weight) $\mathbf{f}_w$, i.e. $\mathbf{f} = \mathbf{f}_a + \mathbf{f}_w$, and similarly for the lagrangian forces $\mathbf{p} = \tilde{\mathbf{C}}^*\mathbf{f}$ (see equ. 3.2,b), i.e.

$$\mathbf{p} = \mathbf{p}_a + \mathbf{p}_w \tag{3.7,a}$$

where we must put:

$$\mathbf{p}_a = \tilde{\mathbf{C}}^*\mathbf{f}_a\,; \quad \mathbf{p}_w = \tilde{\mathbf{C}}^*\mathbf{f}_w. \tag{3.7b,c}$$

The design dependent mass forces $\mathbf{p}_w$ can be expressed in terms of the design variables by the matrix equation [7, 18]:

$$\mathbf{p}_w = \mathbf{W}\mathbf{r}. \tag{3.8}$$

The matrix **W** (*mass force matrix*) depends on the node layout. Linearity in equ. (3.8) is consistent with the hypotheses 2.c,d.

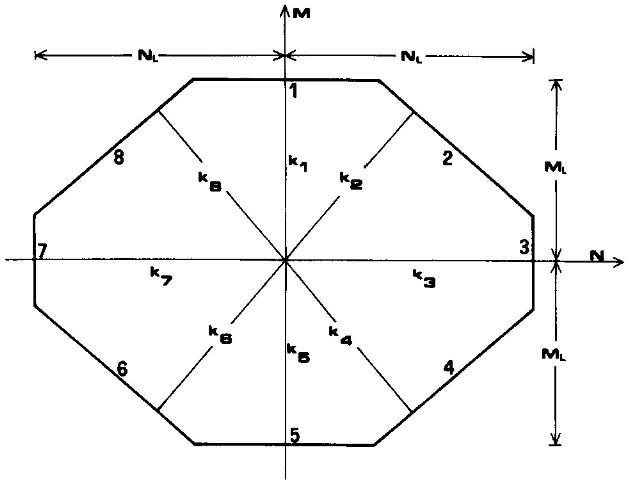

Fig. 1.

For practical reasons, some *technological constraints* [7, 18] can arise. It may be convenient in fact to collect the n finite elements in $m < n$ groups, each being composed of elements whose plastic resistances are in some way linked. In other words in each group the design variables are not $n't$ (n' = number of elements in the group), but only t. If we call $\boldsymbol{\rho}_\mu$ the independent design variable t-vector of the μth group ($\mu = 1,2,\ldots,m$), a matrix relation exists between $\boldsymbol{\rho}_\mu$ and the dependent design variables, such as:

$$\mathbf{r}_{(\mu)} = \mathbf{T}_\mu \boldsymbol{\rho}_\mu + \mathbf{r}^\circ_\mu, \quad (\mu = 1,2,\ldots,m)$$

with $\boldsymbol{\rho}_\mu \geq \mathbf{0}$ and $\mathbf{r}^\circ_\mu$ fixed. In this vector equation we have to take $\tilde{\mathbf{r}}_{(\mu)} = [\tilde{\mathbf{r}}_{\nu_1}, \tilde{\mathbf{r}}_{\nu_2}, \ldots \tilde{\mathbf{r}}_{\nu_{n'}}]$, the ν's being n' distinct values of the index i which marks the n elements. Since $\tilde{\mathbf{r}} = [\tilde{\mathbf{r}}_{(1)}, \tilde{\mathbf{r}}_{(2)}, \ldots, \tilde{\mathbf{r}}_{(m)}]$, putting now $\tilde{\boldsymbol{\rho}} = [\tilde{\boldsymbol{\rho}}_1, \tilde{\boldsymbol{\rho}}_2, \ldots, \tilde{\boldsymbol{\rho}}_m]$, $\tilde{\mathbf{r}}^\circ = [\tilde{\mathbf{r}}^\circ_{(1)}, \tilde{\mathbf{r}}^\circ_{(2)}, \ldots, \tilde{\mathbf{r}}^\circ_{(m)}]$, and $\mathbf{T} = \text{diag}\,[\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_m]$, we can collect the above technological constraints in an equivalent single matrix equation:

$$\mathbf{r} = \mathbf{T}\boldsymbol{\rho} + \mathbf{r}^\circ. \tag{3.9}$$

The vector $\mathbf{r}^\circ$ defines the *minimal design*, i.e. the design obtained giving all the plastic resistances the relevant minimum values ($\boldsymbol{\rho} = \mathbf{0}$).

The $mt \times nt$-matrix $\mathbf{T}$ (*technological matrix*) may be in particular [7]:

an identity matrix, if $m = n$, and then $\mathbf{r} = \boldsymbol{\rho} + \mathbf{r}^\circ$ (no technological constraints);

a boolian matrix, if the yielding polyhedra in each group remain equal to each other in the design growing process;

a vector, if $m = t = 1$, and in this case the design problem identifies with a limit analysis problem.

4. THE OPTIMIZATION PROBLEM

Starting from the two limit analysis theorems, two distinct primal formulations of the optimum plastic design can be derived [7]. We consider now the statical theorem only.

The overall cost (2.2), taking into account equs. (2.4), (3.9), becomes:

$$\Phi = \tilde{\mathbf{l}}\mathbf{a} + \tilde{\mathbf{c}}\mathbf{r}^\circ + \tilde{\mathbf{c}}\mathbf{T}\boldsymbol{\rho}, \tag{4.1}$$

where the *gradient cost vector* c, given by:

$$\mathbf{c} = \mathbf{Bl} \tag{4.2}$$

has been introduced.

The equilibrium equations (3.2,a) with the aid of the equs. (3.7,a), (3.8) and (3.9) and of the definition (4.2), transform into:

$$\tilde{\mathbf{C}}\boldsymbol{\sigma} - \mathbf{WT}\boldsymbol{\rho} = \mathbf{p}^\circ, \quad (\mathbf{p}^\circ = \mathbf{p}_a + \mathbf{Wr}^\circ), \tag{4.3a,b}$$

and $\mathbf{p}^\circ$ is the vector of the loads relative to the minimal design.

The yielding functions (or plastic potentials) (2.1), using equs. (3.5), (3.9) and (4.2), can be written:

$$\boldsymbol{\varphi} = \tilde{\mathbf{N}}\boldsymbol{\sigma} - \mathbf{VT}\boldsymbol{\varphi} - \mathbf{k}^\circ, \quad (\mathbf{k}^\circ = \mathbf{Vr}^\circ), \tag{4.4a,b}$$

$\mathbf{k}^\circ$ being the plastic resistance vector of the minimal design.

Now we can state the design optimization problem in the following way:

$$\textit{minimize } \Phi(\boldsymbol{\rho}) = \tilde{\mathbf{c}}\mathbf{T}\boldsymbol{\rho} \tag{4.5,a}$$

subject to:

$$-\tilde{\mathbf{C}}\boldsymbol{\sigma} + \mathbf{WT}\boldsymbol{\rho} + \mathbf{p}^\circ = \mathbf{0}, \text{(equilibrium)} \tag{4.5,b}$$

$$\tilde{\mathbf{N}}\boldsymbol{\sigma} - \mathbf{VT}\boldsymbol{\rho} - \mathbf{k}^\circ \leq \mathbf{0}, \text{(conformity)} \tag{4.5,c}$$

$$-\boldsymbol{\rho} \leqslant \mathbf{0}, \text{(design positivity)}. \tag{4.5,d}$$

This is a linear programming problem whose objective function $\Phi(\boldsymbol{\rho})$ differs from the total cost (4.1) by a constant term only. A side vector equation of the problem (4.5) is:

$$\delta\mathbf{k} \equiv \mathbf{k} - \mathbf{k}^\circ = \mathbf{VT}\boldsymbol{\rho}. \tag{4.5,e}$$

It is obtained from (3.5) with the aid of (3.9), (4.2) and (4.4,b), and is to be used after a design $\boldsymbol{\rho}$ has been determined.

If we call *statically admissible* a design $\boldsymbol{\rho}$, such that *a stress vector* $\boldsymbol{\sigma}$ *could be associated with it so that equilibrium, conformity and design positivity conditions* (i.e. conditions (4.5 b,c,d) *are satisfied,* the optimization problem (4.5) can be phrased in common words:

PRIMAL DESIGN PROBLEM: *Among all statically admissible designs, find one of minimum cost.*

Following a classical path (see e.g. [23, 24]), we write the optimality conditions for the solution(s) of the problem (4.5) constructing the lagrangian function:

$$\Theta(\boldsymbol{\rho},\boldsymbol{\sigma},\mathbf{x},\mathbf{y},\mathbf{z}) = \Phi(\boldsymbol{\rho}) + \tilde{\mathbf{x}}[-\tilde{\mathbf{C}}\boldsymbol{\sigma} + \mathbf{WT}\boldsymbol{\rho} + \mathbf{p}^\circ] + \tilde{\mathbf{y}}[\tilde{\mathbf{N}}\boldsymbol{\sigma} - \mathbf{VT}\boldsymbol{\rho} - \mathbf{k}^\circ] - \tilde{\mathbf{z}}\boldsymbol{\rho}, \tag{4.6}$$

where x, y, z are some lagrangian vector variables (x has as many free entries as there are equilibrium equations, while the *ns*-vector y and the *mt*-vector z are sign constrained, i.e. $\mathbf{y} \geqslant \mathbf{0}$, and $\mathbf{z} \geqslant \mathbf{0}$), and then differentiating with respect to its arguments. Hence the necessary (and sufficient) conditions for the solution(s) $\boldsymbol{\rho}^*$ are:

$$-\tilde{\mathbf{C}}\boldsymbol{\sigma}^* + \mathbf{WT}\boldsymbol{\rho}^* + \mathbf{p}^\circ = \mathbf{0} \quad (4.7,a)$$
$$\tilde{\mathbf{N}}\boldsymbol{\sigma}^* - \mathbf{VT}\boldsymbol{\rho}^* - \mathbf{k}^\circ \leqslant \mathbf{0} \quad (4.7,b)$$
$$-\boldsymbol{\rho}^* \leqslant \mathbf{0} \quad (4.7,c)$$
$$\mathbf{Ny}^* - \mathbf{Cx}^* = \mathbf{0} \quad (4.8,a)$$
$$\tilde{\mathbf{T}}\mathbf{c} + \tilde{\mathbf{T}}\tilde{\mathbf{W}}\mathbf{x}^* - \tilde{\mathbf{T}}\tilde{\mathbf{V}}\mathbf{y}^* - \mathbf{z}^* = \mathbf{0} \quad (4.8,b)$$
$$\mathbf{y}^* \geqslant \mathbf{0} \quad (4.8,c)$$
$$\mathbf{z}^* \geqslant \mathbf{0} \quad (4.8,d)$$
$$\tilde{\mathbf{y}}^*[\tilde{\mathbf{N}}\boldsymbol{\sigma}^* - \mathbf{VT}\boldsymbol{\rho}^* - \mathbf{k}^\circ] = 0 \quad (4.9,a)$$
$$\tilde{\mathbf{z}}^*\boldsymbol{\rho} = 0, \quad (4.9,b)$$

$\boldsymbol{\sigma}^*, \mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*$ being some vectors associated with $\boldsymbol{\rho}^*$.The first three conditions (4.7) say that the /an optimum design is a statically admissible one. Comparing (4.8a,c) with (3.1,a) and (3.4a,b) we can qualify $\mathbf{x}^*$ and $\mathbf{y}^*$ as velocity and strain rate intensity vectors respectively (for a detailed discussion see [7, 18, 19]), i.e.

$$\dot{\alpha}\mathbf{x}^* = \dot{\mathbf{u}}^*, \text{ and } \dot{\alpha}\mathbf{y}^* = \dot{\boldsymbol{\lambda}}^*, \text{ with } \dot{\alpha} > 0,$$

and the equ. (4.8,a) appears to be the compatibility vector equation, while the condition (4.9,a) becomes identical to (3.4,c). Therefore, the set of conditions (4.8)–(4.9), putting $\boldsymbol{\psi}^* = -\dot{\alpha}\mathbf{z}^*$, transform into:

$$\mathbf{N}\dot{\boldsymbol{\lambda}}^* - \mathbf{C}\dot{\mathbf{u}}^* = \mathbf{0} \quad (4.10)$$
$$\boldsymbol{\psi}^* \equiv \tilde{\mathbf{T}}\tilde{\mathbf{V}}\dot{\boldsymbol{\lambda}}^* - \tilde{\mathbf{T}}\tilde{\mathbf{W}}\dot{\mathbf{u}}^* - \dot{\alpha}\tilde{\mathbf{T}}\mathbf{c} \leqslant \mathbf{0} \quad (4.11)$$
$$\dot{\boldsymbol{\lambda}}^* \geqslant \mathbf{0}, \tilde{\boldsymbol{\varphi}}^*\dot{\boldsymbol{\lambda}}^* = 0, \quad (4.12,a,b)$$
$$\tilde{\boldsymbol{\psi}}^*\boldsymbol{\rho}^* = 0, \quad (4.13)$$

with the side vector equation:

$$\dot{\boldsymbol{\epsilon}}^* = \mathbf{N}\dot{\boldsymbol{\lambda}}^*. \quad (4.14)$$

In other words, *an optimum design* $\boldsymbol{\rho}^*$ *is characterized as that statically admissible design with which it is possible to associate a compatible strain rate vector satisfying the flow-law rules* (equs. (4.12a,b), (4.14)) *as well as the conditions* (4.11) *and* (4.13). This general statement covers the results obtained elsewhere in particular cases (see e.g. [7, 11–13, 18, 20]).

The dual problem associated with (4.5) has the function (4.6) as objective function, which is to be maximized under the conditions (4.10), (4.11), (4.12,a) (see e.g. [23, 24]). Taking into account these conditions and putting $\Psi = \dot{\alpha}\ \Theta$, we have the problem:

$$\textit{maximize } \Psi(\boldsymbol{\lambda},\dot{\mathbf{u}}) = \tilde{\mathbf{p}}^\circ\dot{\mathbf{u}} - \tilde{\mathbf{k}}^\circ\dot{\boldsymbol{\lambda}} \quad (4.15,a)$$

subject to:

$$\mathbf{N}\dot{\boldsymbol{\lambda}} - \mathbf{C}\dot{\mathbf{u}} = \mathbf{0} \text{ (compatibility)} \quad (4.15,b)$$
$$\boldsymbol{\psi} \equiv \tilde{\mathbf{T}}\tilde{\mathbf{V}}\dot{\boldsymbol{\lambda}} - \tilde{\mathbf{T}}\tilde{\mathbf{W}}\dot{\mathbf{u}} - \dot{\alpha}\mathbf{Tc} \leqslant \mathbf{0}, \text{ (uniformity)} \quad (4.15,c)$$
$$-\dot{\boldsymbol{\lambda}} \leqslant \mathbf{0}, \text{ (dissipation potivity)} \quad (4.15,d)$$

the scalar $\dot{\alpha}$ being an arbitrary positive constant. A side vector equation of the problem above is:

$$\dot{\boldsymbol{\epsilon}} = \mathbf{N}\dot{\boldsymbol{\lambda}}, \tag{4.15,e}$$

which is to be used after a deformation rate pattern has been determined.

If we call *geometrically admissible* a strain rate intensity vector $\dot{\boldsymbol{\lambda}}$, such that a *velocity vector* $\dot{\mathbf{u}}$ *could be associated with it so that compatibility, uniformity and dissipation positivity conditions* (i.e. conditions (4.15b,c,d)) *are satisfied*, the optimization problem (4.15) can be phrased in common words:

DUAL DESIGN PROBLEM: *Among all geometrically admissible strain rate intensities, find one which, with reference to the minimal design, maximizes the difference between the power of the external forces and the dissipation rate.*

The duality theorem[23, 24] asserts that a solution $\dot{\mathbf{u}}^*$, $\dot{\boldsymbol{\lambda}}^*$ of the dual problem implies the existence of some $\boldsymbol{\rho}^*$, $\boldsymbol{\sigma}^*$, which solve the primal one, and that the optimal values of the two objective functions are the same, i.e.

$$\dot{\alpha}\tilde{\mathbf{c}}\mathbf{T}\boldsymbol{\rho}^* = \tilde{\mathbf{p}}^\circ\dot{\mathbf{u}}^* - \tilde{\mathbf{k}}^\circ\dot{\lambda}^*.$$

This gives the meaning of the constant $\dot{\alpha}$ (power per unit cost).

5. GEOMETRICAL DESCRIPTION OF OPTIMALITY CONDITIONS

For the aske of simplitity, let the body forces and the technological constraints be momentarily absent. The optimality conditions (4.7)–(4.9), written in the simpler form, can be collected in the two following sets (the stars being omitted):

$$\tilde{\mathbf{C}}\boldsymbol{\sigma} - \mathbf{p}_a = \mathbf{0} \tag{5.1,b}$$

$$\boldsymbol{\varphi} \equiv \tilde{\mathbf{N}}\boldsymbol{\sigma} - \mathbf{V}\boldsymbol{\rho} - \mathbf{k}^\circ \leqslant \mathbf{0} \tag{5.1,b}$$

$$\dot{\boldsymbol{\epsilon}} = \mathbf{N}\dot{\boldsymbol{\lambda}} \tag{5.1,c}$$

$$\tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{\lambda}} = 0 \tag{5.1,d}$$

$$\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0} \tag{5.1,e}$$

$$\mathbf{N}\dot{\boldsymbol{\lambda}} - \mathbf{C}\dot{\mathbf{u}} = \mathbf{0} \tag{5.2,a}$$

$$\boldsymbol{\psi} \equiv \tilde{\mathbf{V}}\dot{\boldsymbol{\lambda}} - \dot{\alpha}\mathbf{c} \leqslant \mathbf{0} \tag{5.2,b}$$

$$\delta\mathbf{k} = \mathbf{V}\boldsymbol{\rho} \tag{5.2,c}$$

$$\tilde{\boldsymbol{\psi}}\boldsymbol{\rho} = 0 \tag{5.2,d}$$

$$\boldsymbol{\rho} \geqslant \mathbf{0}. \tag{5.2,e}$$

Between the two sets a strong similarity exists.† The conditions (b)... (c) of the first set, which express the *conformity* requirements for the stresses $\boldsymbol{\sigma}$ and the strain rates $\dot{\boldsymbol{\epsilon}}$, have their counterparts in the relevant conditions of the second set, which express in turn what we can call the *uniformity* requirements for the strain rate intensities $\dot{\boldsymbol{\lambda}}$ and the plastic resistance increments $\delta\mathbf{k}$. More precisely, besides a (convex *yielding*) surface in the $\boldsymbol{\sigma}$-space and a consistent rule (*plastic flow-law*) for the plastic deformation rates $\dot{\boldsymbol{\epsilon}}$ (first set), we can envisage a (convex *design growing*) surface in the $\dot{\boldsymbol{\lambda}}$-space and a consistent rule (*design grow-law*) for the plastic resistance increments $\delta\mathbf{k}$ (second set).

A correspondance is then established between the $\boldsymbol{\sigma}$-space and the $\dot{\boldsymbol{\lambda}}$-space (see Figs. 2a,b).

†It is always possible to define the design variables in such a way that the columns of the matrix $\mathbf{V}_T = \mathbf{VT}$ are unit vectors.

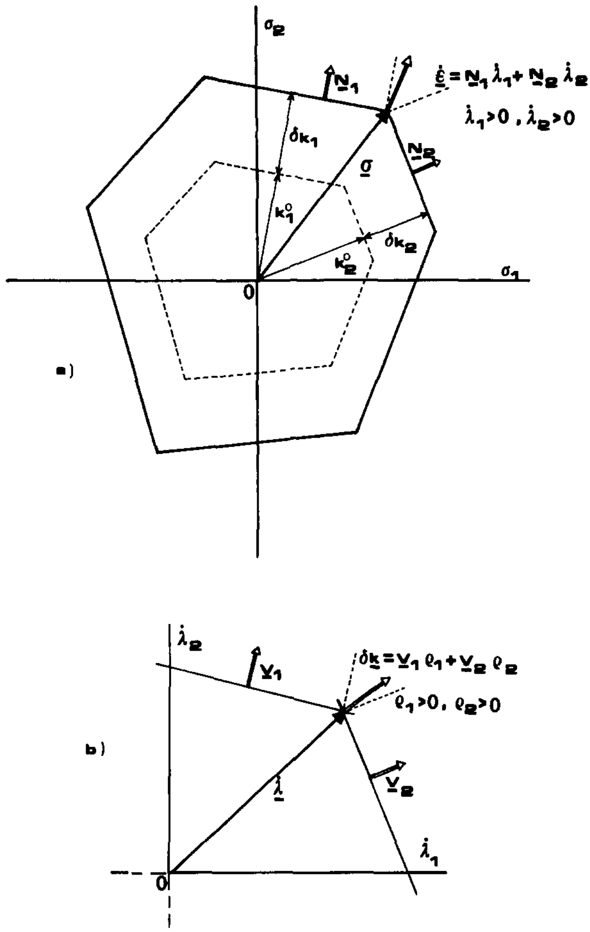

Fig. 2.

In the $\boldsymbol{\sigma}$-space

(a) A (convex) *yielding surface* is given defining the range of $\boldsymbol{\sigma}$-points where the *plastic potentials* $\boldsymbol{\varphi}(\boldsymbol{\sigma}) \leq \mathbf{0}$;

(b) A law is defined for *plastic yielding* by which the strain rate vector $\dot{\boldsymbol{\epsilon}}$, associated with a given $\boldsymbol{\sigma}$, is determined, i.e.

(i) $\dot{\boldsymbol{\epsilon}} = \mathbf{0}$, if the $\boldsymbol{\sigma}$-point is inside the plastic yielding surface ($\dot{\boldsymbol{\lambda}} = \mathbf{0}$, $\boldsymbol{\varphi} < \mathbf{0}$), and
(ii) $\dot{\boldsymbol{\epsilon}} \neq \mathbf{0}$, if the $\boldsymbol{\sigma}$-point is on the yielding surface, and complies with the external normal rule (in the Koiter sense [25, 21]).

In the $\dot{\boldsymbol{\lambda}}$-space

(a) A (convex) *design growing surface* is given defining the range of $\dot{\boldsymbol{\lambda}}$-points where the *design potentials* $\boldsymbol{\psi}(\dot{\boldsymbol{\lambda}}) \leq \mathbf{0}$;

(b) A law is defined for *design growing* by which the plastic resistance increment vector $\delta\mathbf{k}$, associated with a given $\dot{\boldsymbol{\lambda}}$, is determined, i.e.

(i) $\delta\mathbf{k} = \mathbf{0}$, if the $\dot{\boldsymbol{\lambda}}$-point is inside the design growing surface ($\boldsymbol{\rho} = 0$, $\boldsymbol{\psi} < \mathbf{0}$), and
(ii) $\delta\mathbf{k} \neq \mathbf{0}$, if the $\dot{\boldsymbol{\lambda}}$-point is on the growing surface, and complies with the external normal rule (in an extended Koiter sense).

Uniformity requirements can be expressed as well by the following:

STATEMENT FOR OPTIMUM PLASTIC DESIGN: *If the $\dot{\boldsymbol{\lambda}}$-point is inside the design growing surface, the maximum economy is in keeping the minimal plastic resistances; while if the $\dot{\boldsymbol{\lambda}}$-point is on the design growing surface, the maximum economy is obtained by increasing the plastic resistance vector by an increment* $\delta\mathbf{k}$ *lying along the external normal to the growing surface in the $\dot{\boldsymbol{\lambda}}$-point* (see Fig. 2(b) where only the $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$-plane of the $\dot{\boldsymbol{\lambda}}$-space is considered).

In the primal (dual) formulation of the optimum plastic design only the first two constraints of the first (second) set, and the last constraint of the second (first) set are retained, but in the limit, when the solution is reached, all the other constraints of both sets are satisfied too, and as a result the solution complies with all the requirements of equilibrium (5.1,a), conformity (5.1b–e),compatibility (5.2,a) and uniformity (5.2b–e).

The relations (5.1), (5.2) appear to be a set of equations governing a *nonlinear elastic problem*, whose constitutive equations are described by (5.1b–e) and (5.2b–e). In this sense these latter equations are to be considered as the piece-wise linear version of those constitutive equations derived by Marcal and Prager [11], and Prager and Shield [12] in their theory of the cost gradient principle.

If the restrictive hypothesis of no body forces and no technological constraints is now removed, the optimality conditions become:†

$$\tilde{\mathbf{C}}\boldsymbol{\sigma} - \mathbf{W}_T\boldsymbol{\rho} - \mathbf{p}^\circ = \mathbf{0} \qquad (5.3,a)$$
$$\boldsymbol{\varphi} \equiv \tilde{\mathbf{N}}\boldsymbol{\sigma} - \mathbf{V}_T\boldsymbol{\rho} - \mathbf{k}^\circ \leqslant \mathbf{0} \qquad (5.3,b)$$
$$\dot{\boldsymbol{\epsilon}} = \mathbf{N}\dot{\boldsymbol{\lambda}} \qquad (5.3,c)$$
$$\tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{\lambda}} = 0 \qquad (5.3,d)$$
$$\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0} \qquad (5.3,e)$$
$$\mathbf{N}\dot{\boldsymbol{\lambda}} - \mathbf{C}\dot{\mathbf{u}} = \mathbf{0} \qquad (5.4,a)$$
$$\boldsymbol{\psi} \equiv \tilde{\mathbf{V}}_T\dot{\boldsymbol{\lambda}} - \tilde{W}_T\dot{\mathbf{u}} - \dot{\alpha}\mathbf{c}_T \leqslant \mathbf{0} \qquad (5.4,b)$$
$$\delta\mathbf{k} = \mathbf{V}_T\boldsymbol{\rho} \qquad (5.4,c)$$
$$\tilde{\boldsymbol{\psi}}\boldsymbol{\rho} = 0 \qquad (5.4,d)$$
$$\boldsymbol{\rho} \geqslant \mathbf{0}, \qquad (5.4,e)$$

where the following definitions have been established:

$$\mathbf{V}_T = \mathbf{VT};\ \mathbf{W}_T = \mathbf{WT};\ \mathbf{c}_T = \tilde{\mathbf{T}}\mathbf{c}. \qquad (5.5,a\text{–}c)$$

The second term in (5.4,b) plays the same role as the second one in (5.3,b) or (5.1,b). In these latter two, a vector $\boldsymbol{\rho}$ different from zero implies that the faces of the yielding polyhedra have distances from the origin $\boldsymbol{\sigma} = \mathbf{0}$ greater than $\mathbf{k}^\circ$. Similarly a vector $\dot{\mathbf{u}}$ different from zero in (5.4,b) causes the distances from the origin $\dot{\boldsymbol{\lambda}} = \mathbf{0}$ of the faces of the design growing polyhedra to differ from $\dot{\alpha}\mathbf{c}_T$.

About the two sets (5.3), (5.4) we can repeat what has been said with reference to the two sets (5.1), (5.2). As it is known [7, 18, 19], the structural optimization problem may in this case be unsolvable when mass forces prevail on service forces.

6. APPLICATION

As a simple application we consider the plane pin-jointed structure of Fig. 3, having equal contour bars. The plastic resistances of the bars 1 to 4 are equal to each other and they all are

†See footnote on page 546.

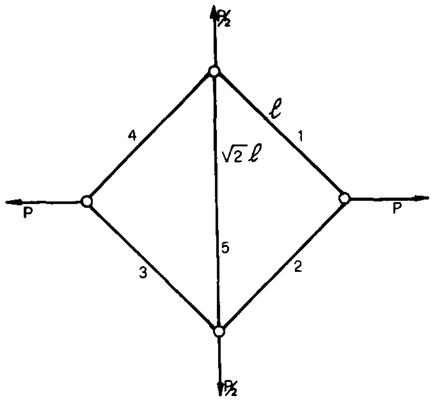

Fig. 3.

different from that of the 5th. Putting:

$$\begin{bmatrix} k_1^+ \\ k_1^- \\ \cdot \\ \cdot \\ k_4^+ \\ k_4^- \\ k_5^+ \\ k_5^- \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} + \begin{bmatrix} \bar{k}_1 \\ \bar{k}_1 \\ \cdot \\ \cdot \\ \bar{k}_1 \\ \bar{k}_1 \\ \bar{k}_2 \\ \bar{k}_2 \end{bmatrix}, \quad (\rho_1, \rho_2) \geq 0$$

we have only two design variables, say ρ_1 and ρ_2, and the minimal design has the plastic resistance vector $\mathbf{k}^\circ$ given by the second addend of the matrix relation above. In the $\boldsymbol{\sigma}$-space, the stress point is readily determined as that of coordinates $\sigma_1 = \cdots = \sigma_4 = P/\sqrt{2}$, $\sigma_5 = -P/2$, and it can be

(i) inside the yielding polyhedron of the minimal design, or

(ii) outside it. The cost function is $\Phi = l(4\rho_1 + \sqrt{2}\rho_2)$, and the design growing potentials are only the following two:

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 1\,1 \cdot\cdot\, 1\,1\,0\,0 \\ 0\,0 \cdot\cdot\, 0\,0\,1\,1 \end{bmatrix} \begin{bmatrix} \dot{\lambda}_1^+ \\ \dot{\lambda}_1^- \\ \cdot \\ \cdot \\ \dot{\lambda}_4^+ \\ \dot{\lambda}_4^- \\ \dot{\lambda}_5^+ \\ \dot{\lambda}_5^- \end{bmatrix} - \dot{\alpha} \begin{bmatrix} 4l \\ \sqrt{2}l \end{bmatrix},$$

where of course we have to take $\dot{\lambda}_1^+ = \cdots = \dot{\lambda}_4^+$ and $\dot{\lambda}_1^- = \cdots = \dot{\lambda}_4^-$, and the design growing surface is formed by the two planes:

$$\dot{\lambda}_1^+ + \dot{\lambda}_1^- + \cdots + \dot{\lambda}_4^+ + \dot{\lambda}_4^- = 4l\dot{\alpha}, \text{ or } \dot{\lambda}_1^+ + \dot{\lambda}_1^- = l\dot{\alpha},$$

and

$$\dot{\lambda}_5^+ + \dot{\lambda}_5^- = \sqrt{2}l\dot{\alpha},$$

with $\dot{\alpha}$ arbitrary positive.

The problem is easily controlled looking only at the projection of the $\boldsymbol{\sigma}$-point into the (σ_1, σ_5)-plane (points A_σ in Figs. 4,a–c) and at that of the $\dot{\boldsymbol{\lambda}}$-point into the $(\dot{\lambda}_1^+, \dot{\lambda}_5^-)$-plane (points A_λ in Figs. 5, a–c).

In the first case above, the minimal design itself is the optimum design. Hence no plastic resistance increments are demanded ($\delta \mathbf{k} = \mathbf{0}$) and the plastic strain rates are all zero ($\dot{\boldsymbol{\lambda}} = \mathbf{0}$).

In the second case (stress point outside the yielding surface relative to the minimal design), we

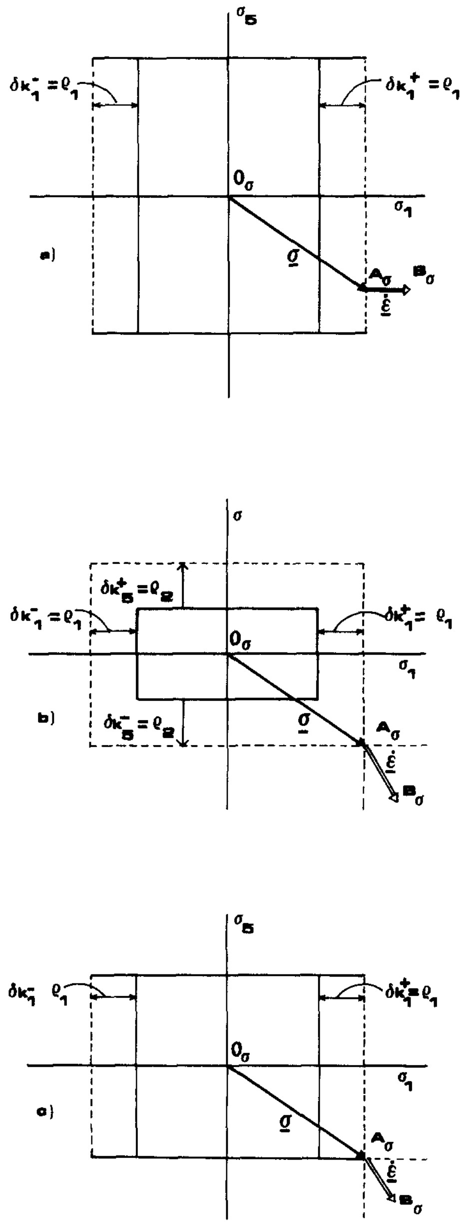

Fig. 4.

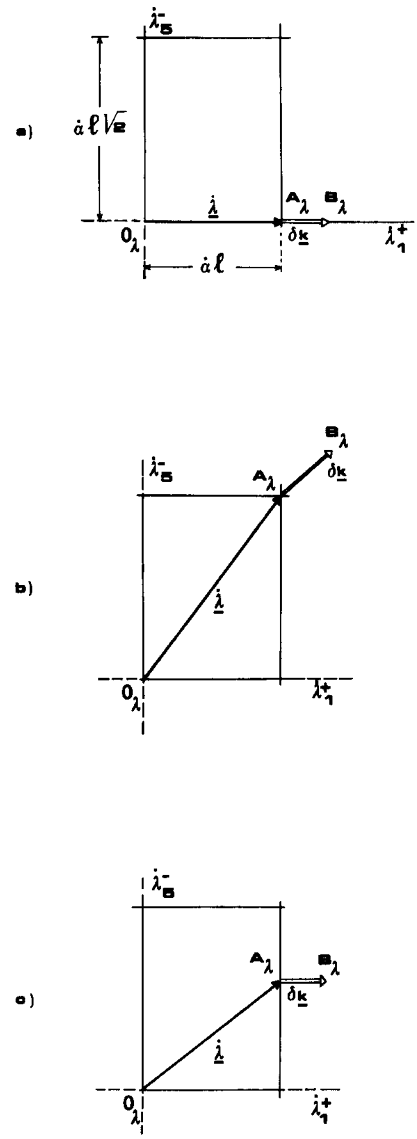

Fig. 5.

can distinguish, for sake of clarity, the following three situations:

(a) The vector $\boldsymbol{\sigma}$ crosses the yielding faces $k_j^+ = \bar{k}_1$ $(j = 1, \ldots, 4)$, (Fig. 4a). These faces translate until the $\boldsymbol{\sigma}$-point stays on them, causing some increment $\delta\hat{\mathbf{k}}$ of the plastic resistance vector (i.e. $\delta\hat{k}_j^+ = \delta k_j^- = \hat{\rho}_1 > 0$, and $\delta\hat{k}_5^+ = \delta\hat{k}_5^- = \hat{\rho}_2 = 0$). The bars 1,2,3 and 4 are at their positive yield point, while the 5th bar is still rigid (i.e. $\dot{\lambda}_j^+ > 0$, $\dot{\lambda}_j^- = 0$, $(j = 1, \ldots, 4)$, and $\dot{\lambda}_5^+ = \dot{\lambda}_5^- = 0$). The vector $\dot{\boldsymbol{\lambda}}$ is completely determined when it is associated (in the sense of the design grow-law, see equs. (5.2b–d)) with the increment vector $\delta\hat{\mathbf{k}}$, which must be along the external normal to the design growing surface in the point $\dot{\boldsymbol{\lambda}}$. From Fig. 5(a) we see that $\dot{\lambda}_j^+ = \dot{\alpha} l$, $\dot{\lambda}_j^- = 0$

($j = 1, \ldots, 4$) and $\dot{\lambda}_5^+ = \dot{\lambda}_5^- = 0$. We check the resulting strain rate vector $\hat{\boldsymbol{\epsilon}}$ is associated with just the vector $\boldsymbol{\sigma}$ (in the sense of the plastic flow-law).

(b) The vector $\boldsymbol{\sigma}$ crosses the yielding faces $k_j^+ = \bar{k}_1$ ($j = 1, \ldots, 4$), and $k_5^- = k_2^-$ (Fig. 4b). These faces translate until the $\boldsymbol{\sigma}$-point stays on them, causing some increment $\delta\hat{\mathbf{k}}$ of the plastic resistance vector (i.e. $\delta\hat{k}_j^+ = \delta\hat{k}_j^- = \hat{\rho}_1 > 0$ ($j = 1, \ldots, 4$), and $\delta\hat{k}_5^+ = \delta\hat{k}_5^- = \rho_2 > 0$). The bars 1,2,3 and 4 are at their positive yield point, and the 5th bar is at its negative yield point (i.e. $\dot{\lambda}_j^+ > 0$, $\dot{\lambda}_j^- = 0$, ($j = 1, \ldots 4$,), and $\dot{\lambda}_5^+ = 0$, $\dot{\lambda}_5^- > 0$). The vector $\dot{\boldsymbol{\lambda}}$ is again completely determined when it is associated with the increment vector $\delta\hat{\mathbf{k}}$, and from Fig. 5(b) we see that $\dot{\lambda}_j^+ = \dot{\alpha}l$, $\dot{\lambda}_j^- = 0$ ($j = 1, \ldots, 4$), and $\dot{\lambda}_5^+ = 0$, $\dot{\lambda}_5^- = \sqrt{2}\dot{\alpha}l$. The resulting strain rate vector $\hat{\epsilon}$ is still associated with the stress vector $\boldsymbol{\sigma}$.

(c) The vector $\boldsymbol{\sigma}$ crosses the yielding faces $k_j^+ = k_1^-$ ($j = 1, \ldots, 4$) and touches the face $k_5^- = \bar{k}_2$ (Fig. 4c). The increment vector $\delta\hat{\mathbf{k}}$ is like that in case (a), but all the bars are in the limit state as in case (b). The vector $\dot{\boldsymbol{\lambda}}$ is now not completely determined because any $\dot{\boldsymbol{\lambda}}$ having the components $\dot{\lambda}_j^+ > 0$, $\dot{\lambda}_j^- = 0$, ($j = 1, \ldots, 4$), and $\dot{\lambda}_5^+ = 0$, $\dot{\lambda}_5^- = 0$, can be associated with the increment $\delta\hat{\mathbf{k}}$ above (see Fig. 5c).

CONCLUSIONS

We have studied a way of finding an optimum plastic structure, as a structure of minimum cost. Since this cost is linearly dependent on design variables, an optimization problem in linear programming is encountered.

In the domain of discrete models, the class of structures considered in the present theory is sufficiently wide and it includes frames [7], discs [18] and other types of structures giving rise to plane stress problems [19]. Continuous structures [11, 12] could be treated too, provided that a straightforward transition is made from the present discrete theory to a proper continuous one.

Nonlinear problems, i.e. problems of the kind treated in [13]—cost nonlinearly dependent on the design variables—are not covered in this paper, but an extension of the present theory to cover these nonlinear cases is an attractive proposition to be developed in future work.

Optimality conditions, in the geometrical version given here, introduce some apparently new concepts (growing process, grow-law, design growing surface, ...) which demand a deeper and more exhaustive discussion, possibly with reference to a continuous model.

From the computational point of view, a paper is in preparation, which concerns the optimum design of plane frames taking into account, in addition to the bending moment, the generalized normal stress.

REFERENCES

1. A. G. M. Michell, The limits of economy in frame structure, *Phil. Mag.* **8**, 589–597 (1954).
2. C. Massonet and M. Save, *Calcul plastique des constructions: structures á un paramètre.* C. B. L. I. A. Pub., Brussels (1967).
3. C. Y. Sheu and W. Prager, Recent developments in optimal structural design, *Appl. Mech. Rev.* **10**, 985–992 (1968).
4. R. H. Bigelow and E. H. Gaylord, Minimum weight of plastically designed steel frames, *J. of Structural Div., ASCE* **93**, ST 6 (1967).
5. M. Z. Cohn and D. E. Grierson, Optimal design of reinforced concrete beams and frames, *8th IABSE Congress publ.* pp. 215–226 (1968).
6. D. E. Grierson and M. Z. Cohn, A general formulation of the optimal frame problem, *J. Appl. Mech. Trans. ASME* **70**, 356–360 (1970).
7. G. Maier, R. Srinivasan and M. A. Save, On limit design of frames using linear programming, *Proc. Int. Symp. on Computer-Aided Structural Design*, Warwick, Coventry, July 1972. Vol. 1, pp. A2.32–59, P. Pé regrinus (1972).
8. W. Prager, Minimum weight design of portal frames, *J. Engng Mech. Div., Proc. ASCE* **82**, 1073 (1956).
9. L. G. Vargo, Nonlinear minimum weight design of planar structures, *J. Aeronaut. Sci.* **23**, 956–961 (1956).
10. R. T. Shield, Optimum design methods for structures, in *Plasticity*, edited by E. H. Lee and P. S. Symonds, pp. 580–591. Pergamon Press (1960).

11. P. V. Marcal and W. Prager, A method of optimal plastic design, *J. de Mecanique* **3**, 509–530 (1964).
12. W. Prager and R. T. Shield, A general theory of optimal plastic design, *J. of Appl. Mech. Trans. ASME* **34**, 184–186 (1967).
13. H. S. Y. Chan, Mathematical programming in optimal plastic design, *Int. J. Solids and Struct.* **4**, 885–895 (1968).
14. G. I. N. Rozvany and R. S. Adidam, Recent advances in optimal plastic design, *Proc. Int. Symp. on Foundations of Plasticity*, edited by A. Sawczuk. Noordhoff. Leyden (1973).
15. R. Kalaba, Design of minimum-weight structures for given reliability and cost, *J. of Aerospace Science* **29**, 355–356 (1962).
16. A. C. Palmer, Optimum structure design by dynamic programming, *J. of the Structural Div., Proc. ASCE* **94**, ST 8, 1887–1904 (1968).
17. D. J. Sheppard and A. C. Palmer, Optimum design of transmission towers by dynamic programming, *Computers and Structures* **2**, 455–468 (1972).
18. G. Maier and A. Zavelani-Rossi, A finite element approach to optimal design of plastic discs, Part. I, Theory, Istituto di Scienza e Tecnica delle Costruzioni, Politecnico di Milano, pub. no. 482 (1970).
19. D. Benedetti, G. Maier and A. Zavelani-Rossi, A finite element approach to optimal design of plastic structures in plane stresses, *Atti del 1° Congresso Nazionale di Mecannica Teorica ed Applicata*, Vol. 2°, parte 1a, pp. 167–211, Udine (1971).
20. D. C. Drucker and R. T. Shield, Design for minimum weight, *Proc. Int. Congr. Appl. Mech., Brussels, Book* 5, pp. 212–222 (1956).
21. G. Maier, Linear flow-laws of elastoplasticity: a unified general approach, *Accademia Nazionale dei Lincei, Rendiconti Classe Scienze fisiche matematiche e naturali, Serie* VIII, *vol* XLII, *fasc.* 5, pp. 266–277 (1969).
22. G. Maier, Constrained optimization in elastoplastic analysis, Politecnico di Milano, *Tech. Report No* 4. I. S. T. C. (1972).
23. G. B. Dantzig, *Linear programming and extensions.* Princeton University Press (1963).
24. O. L. Mangasarian, *Nonlinear programming.* McGraw-Hill (1969).
25. W. T. Koiter, General theorems for elastic–plastic solids, in *Progress in Solid Mechanics* I, edited by I. N. Sneddon and R. Hill. North-Holland Publ. Co., Amsterdam (1964).